M.I.T Media Laboratory Perceptual Computing Section Technical Report No. 175
Appeared: SPIE Intelligent Robots and Computer Vision X: Algorithms and Techniques, Boston, MA, Nov 1991

Markov/Gibbs modeling: Texture and Temperature *

Rosalind W. Picard and Alex P. Pentland

Vision and Modeling Group
The Media Laboratory
Massachusetts Institute of Technology
20 Ames St., Cambridge, MA 02139

Abstract

Markov/Gibbs random fields have been used for posing a variety of computer vision and image processing problems. Many of these problems are then solved using a simulated annealing type of method which involves the varying of the "temperature," a scale parameter for the model. In this paper we analyze the effect of temperature on random field texture patterns. We obtain new results relating structure in the texture co-occurrence matrix to temperature. We also show the existence of multiple "transition temperatures" which delimit regions of different bandwidth in the co-occurrence matrix, and hence can be used to control pattern formation.

1 Introduction

The indirect use of temperature in Markov/Gibbs random fields (MRF)'s for image restoration, compression, segmentation, and classification has been widespread. In these applications it is typically assumed that the MRF is a prior model for the original image, which may have texture. The resulting posterior distribution is often maximized using some variation of simulated annealing, an optimization technique that tries to minimize an energy function while "cooling" the system. The effects of temperature on the convergence of the optimization have been studied in the simulated annealing literature. In this paper we show that the temperature also influences the resulting texture pattern, and consequently, we show how it can be used to control pattern formation.

It is known that a separation into different scales can be achieved by varying temperature. This property of temperature is the key motivation behind the use of simulated annealing for solving large optimization problems. Kirkpatrick *et. al.* [1] state, "the temperature distinguishes classes of rearrangements, so that rearrangements causing large changes in the objective function occur at high temperatures, while the small changes are deferred until low temperatures." In general, textures are characterized by behavior at different scales. It follows that their scale behavior may be linked somehow to the temperature of the model synthesizing the pattern. Marroquin [2] has noticed that temperature is linked to the size and density of clusters in some patterns. This paper establishes some precise new connections between texture structure and temperature.

*This research was made possible in part by National Science Foundation, Grant No. IRI-8719920, by the Rome Air Development Center (RADC) of the Air Force System Command and the Defense Advanced Research Projects Agency (DARPA) under contract No. F30602-89-C-0022.

1.1 Physical temperature analogies

Before proceeding with our new results it is helpful to formulate the following analogy with two physical processes. The analogy comes from chemistry, where temperature is an important control for purifying substances and forming crystals. Two basic principles are at work in this process, both of which are helpful in explaining the behavior of MRF texture formation.

The first principle takes advantage of the different solubilities of substances with respect to temperature. At some high temperature all substances are dissolved. As the temperature is lowered, different components precipitate out of solution and form crystals. Basically, temperature acts as a "control knob" for the separation of different substances [3].

The temperature at which a substance begins a change in state is called a "critical temperature," or in the case of crystallization, the "freezing point." Important changes in physical systems are characterized by their critical temperatures. For example, in purified mercury the critical temperature marks where the metal has zero resistivity and becomes superconducting. Perhaps the most familiar critical points are the temperatures where ice becomes water and where water becomes steam, i.e., the points of *phase transition*.

The second principle involves the formation of a *regular* crystal. At high temperatures, molecules are free to move any distance; they mix easily. At some critical temperature the desired crystalline substance begins to solidify. To obtain a pure regular crystal the lowering of temperature in this critical range must proceed very slowly, spending lots of time where the molecules can make adjustments at small scales. Intuitively, at high temperature the components of the crystal can also move freely over large scale, whereas at low temperature the mobility is less and restricts changes to small scales.

These physical analogies will be related to three components of this research. First, using an analogy between solutions and graylevels, we will see that different "graylevels" precipitate out of a mixture with one another at different temperatures. Second, we relate the notion of critical temperature to "transition temperature" where behavior similar to phase transition occurs in the texture. Finally, it will be seen that the local-interaction random field model produces large-scale ordered structure at low temperature.

2 Background

2.1 Notation

This paper focuses on the discrete Markov/Gibbs random field, defined as follows. Let an image be represented by a finite rectangular $M \times N$ lattice $\mathcal{S}$ with a neighborhood structure $\mathbf{N} = \{\mathcal{N}_s, s \in \mathcal{S}\}$ where $\mathcal{N}_s \subseteq \mathcal{S}$ is the set of sites which are neighbors of the site $s \in \mathcal{S}$. Every site has a graylevel value $x_s \in \Lambda = \{0, 1, \ldots, n-1\}$. Let $\{\mathcal{S}_g, g \in \Lambda\}$ be a partition of the graylevels in $\mathcal{S}$, where $x_s = g$ if $s \in \mathcal{S}_g$. Let $\mathbf{x}$ be the vector $(x_s, 1 \leq s \leq |\mathcal{S}|)$ of all site graylevel values and Ω be the set of all configurations taken by $\mathbf{x}$. A neighborhood structure is said to be *symmetric* if $\forall s, r \in \mathcal{S}$, $s \in \mathcal{N}_r$ if and only if $r \in \mathcal{N}_s$. We assume that the image lattice is periodic to simplify notation; thus, all the textures appear as if the top of the lattice is connected to the bottom, and the left to the right.

For the finite periodic lattice $\mathcal{S}$ with a symmetric neighborhood structure one can define a Gibbs energy, $E(\mathbf{x})$, which depends on the current arrangement of pixel values. There are many ways to define the energy; the choice studied here is the *auto-binomial* energy of [4], which has been shown to produce a variety of natural looking textures [5]. The homogeneous auto-binomial energy is

$$E(\mathbf{x}) = -\sum_{s \in \mathcal{S}} \left(\alpha x_s + \sum_{r \in \mathcal{N}_s} \beta_r x_s x_r \right), \qquad (1)$$

where the model parameters are α, the *external field*, and β_r, the bonding parameters. The β_r control the internal interaction of the texture. A joint probability distribution is assigned to the Gibbs energy yielding the *Gibbs random field*,

$$P(\mathbf{x}) = \frac{1}{Z} \exp\left(-\frac{1}{T} E(\mathbf{x})\right), \qquad (2)$$

where Z is a positive normalizing constant known in the physics literature as a *partition function* and T is the "temperature" of the field.

A texture image corresponds to a particular configuration, $\mathbf{x}$, sampled from this random field. A texture is typically synthesized using a Monte Carlo method such as the Metropolis exchange algorithm. The synthesis process iteratively tries to find the configuration in Ω which maximizes the probability $P(\mathbf{x})$, minimizing the Gibbs energy. In this algorithm, it can be shown that the external field term cancels, leaving only the bonding parameters. In this paper we use the Metropolis exchange algorithm and uniform histogram constraint of [5]; hence, the rest of our formulation considers just the internal interactions.

2.2 Energy in terms of co-occurrences

In [6] it was showed that an equivalent way to write the auto-binomial Gibbs energy (without external field) is

$$E(\mathbf{x}) = -\sum_{g, g' \in \Lambda} g g' \sum_{k=1}^{K} \beta_k m^k(g, g') \qquad (3)$$

where

$$m^k(g, g') = \sum_{s \in \mathcal{S}_g} |\mathcal{N}_s^k \cap \mathcal{S}_{g'}|$$

is known as the "aura measure over $\mathcal{N}_s^k$" since it satisfies the properties of a measure over an "aura set". Intuitively, the aura set is a dilated region surrounding a pixel, and the aura measure indicates how much of each graylevel is contained in that neighboring region. Here, the dilated region is $\mathcal{N}_s^k$, a subneighborhood of the Markov neighborhood,

$$\mathcal{N}_s = \bigcup_{k=1}^{K} \mathcal{N}_s^k, \; \forall s \in \mathcal{S},$$

where $\mathcal{N}_s^k \cap \mathcal{N}_s^l = \emptyset$, unless $l = k$. As reported earlier [6], the aura measure over a set is equal to a sum of the co-occurrences taken over the displacements in that set.

By grouping all of the aura measures into the matrix $\mathbf{A}$, the energy in (3) can be written as

$$E(\mathbf{x}) = -\sum_{k=1}^{K} \beta_k \mathbf{g}^T \mathbf{A}^k \mathbf{g},$$

where $\mathbf{A}^k$ has as elements $m^k(g, g')$. When the model is isotropic then $\beta_k = \beta$ and

$$E(\mathbf{x}) = -\beta \mathbf{g}^T \mathbf{A} \mathbf{g}, where \mathbf{A} = \sum_{k=1}^{K} \mathbf{A}^k. \qquad (4)$$

Let $\mathbf{C}(d)$ be the co-occurrence matrix at displacement d. Then we also have

$$\mathbf{A} = \sum_{s+d \in \mathcal{N}_s} \mathbf{C}(d).$$

Using the relationship between co-occurrence and auto-correlation, we have that the energy is also equivalent to the texture's auto-correlation, formed over just the displacements which occur in the Markov neighborhood $\mathcal{N}_s$ [7]. Thus, the isotropic energy functions are nearly the same as the auto-correlation.

As mentioned, texture is synthesized iteratively while this energy function is minimized. With this formulation, it can be seen that two textures with the same co-occurrences will have the same Gibbs energy (and textures with different co-occurrences may or may not have different energies.) Thus, examining energy alone will not discriminate textures as well as will examining the structure of the aura matrix, $\mathbf{A}$, and hence of the co-occurrence matrices.

3 Texture and Temperature

3.1 Phase and temperature transitions

Several people have suggested the importance of exploring phase transition phenomena for the Markov/Gibbs random field in image processing [8, 9]. The critical point for the particular case of the Ising model with a small positive external field was examined empirically by Carnevali *et. al.* [10]. A phenomenological method was also used by Van Den Bout and Miller for a graph partitioning problem [11] where they showed that most of the optimization occurred around a critical temperature.

However, the methods in physics typically used for finding critical temperatures examine changes in magnetism or spin averages, i.e., first order statistics for an image. When an image histogram is constrained, as in synthesis with a Monte Carlo exchange algorithm, then these simple averages will not work. Alternatively, we could look at second order-statistics. Pickard [12] used a measure based on correlation, i.e. second order statistics, in his study of critical phenomena for the Ising model.

Since correlation of the image is formed as a linear combination of co-occurrences, it is typically not as informative as the co-occurrence statistics for measuring changes in the pattern. Thus, we would expect co-occurrences might be even better for finding critical points. Indeed, we find that the aura matrix exhibits sharp changes corresponding to distinct types of patterns.

In statistical mechanics the critical points are defined to be the zeros of Z, the partition function. When Z is known, the critical temperature(s) can be found analytically. To date, Z is only known for the special case of a binary Ising model with zero external field. For this case there is one critical temperature, $T_c = 2.27$, and it corresponds to the maximum value of the temperature at which the center of the lattice is affected by the boundary conditions [13].

In general Z is unknown and its zeros cannot be determined. Also, one dimension of the lattice needs to go to infinity for a phase transition to exist. Thus, the finite textures modeled here do not have a critical temperature. Nevertheless, the patterns still exhibit abrupt changes in behavior that mimic that of a phase transition. We call the temperature where the behavior changes a *transition temperature* and denote it by T_*.

We now consider the problem of locating the transition temperature T_* for an isotropic MRF. As mentioned, this problem cannot be solved analytically since the partition function, Z, is unknown. It is also difficult to solve by simulation since there are an infinite number of possible parameter combinations to explore. We restrict our attention to a first order isotropic auto-binomial MRF, with $\beta = 1$, and to the consideration of the Gibbs energy and its aura matrix. In [14] we examined this for the binary MRF using the method of specific heat. We review the specific heat method briefly, and then examine the general graylevel case where $n > 2$.

3.1.1 Specific heat

One empirical method for finding a transition temperature is to determine the *specific heat* of the system. In physics the specific heat measures the change of energy with respect to temperature at a given volume or pressure. Simulated annealing researchers have estimated this quantity and found it to have a large value when the state of order of the system was changing. Kirkpatrick *et. al.* [1] interpreted this point as the place where freezing begins and where the process should be cooled the slowest. At an idealized critical temperature, the specific heat would be infinite. For our finite textures, the peaks of the specific heat should correspond to transition temperatures. We review the formulation of this quantity for textures, and show simulations of its computation.

The specific heat is formulated as follows. In statistical mechanics the average energy is

$$< E(T) > = \frac{-d \ln Z}{d(1/k_{\mathbf{B}}T)},$$

where $k_{\mathbf{B}}T$ is the Boltzmann constant. The rate of change of the energy with respect to temperature gives the size of typical fluctuations in the energy at a given T [1],

$$\frac{d < E(T) >}{dT} = \frac{< E(T)^2 > - < E(T) >^2}{k_{\mathbf{B}}T^2}.$$

This quantity is used in physics to measure the specific heat. For texture measurements we normalize by the lattice size and omit the Boltzmann constant, obtaining

$$C(T) = \frac{(< E(T)^2 > - < E(T) >^2)}{T^2|\mathcal{S}|},$$

the specific heat for an MRF sample. Since the Metropolis synthesis algorithm samples with the Boltzmann factor, $\exp(-E/T)$, the evaluation of the mean of the energy at T, $< E(T) >$, is just the sum

$$< E(T) > = \frac{\sum_{i=1}^{i_{eq}} E_i(T)}{i_{eq}},$$

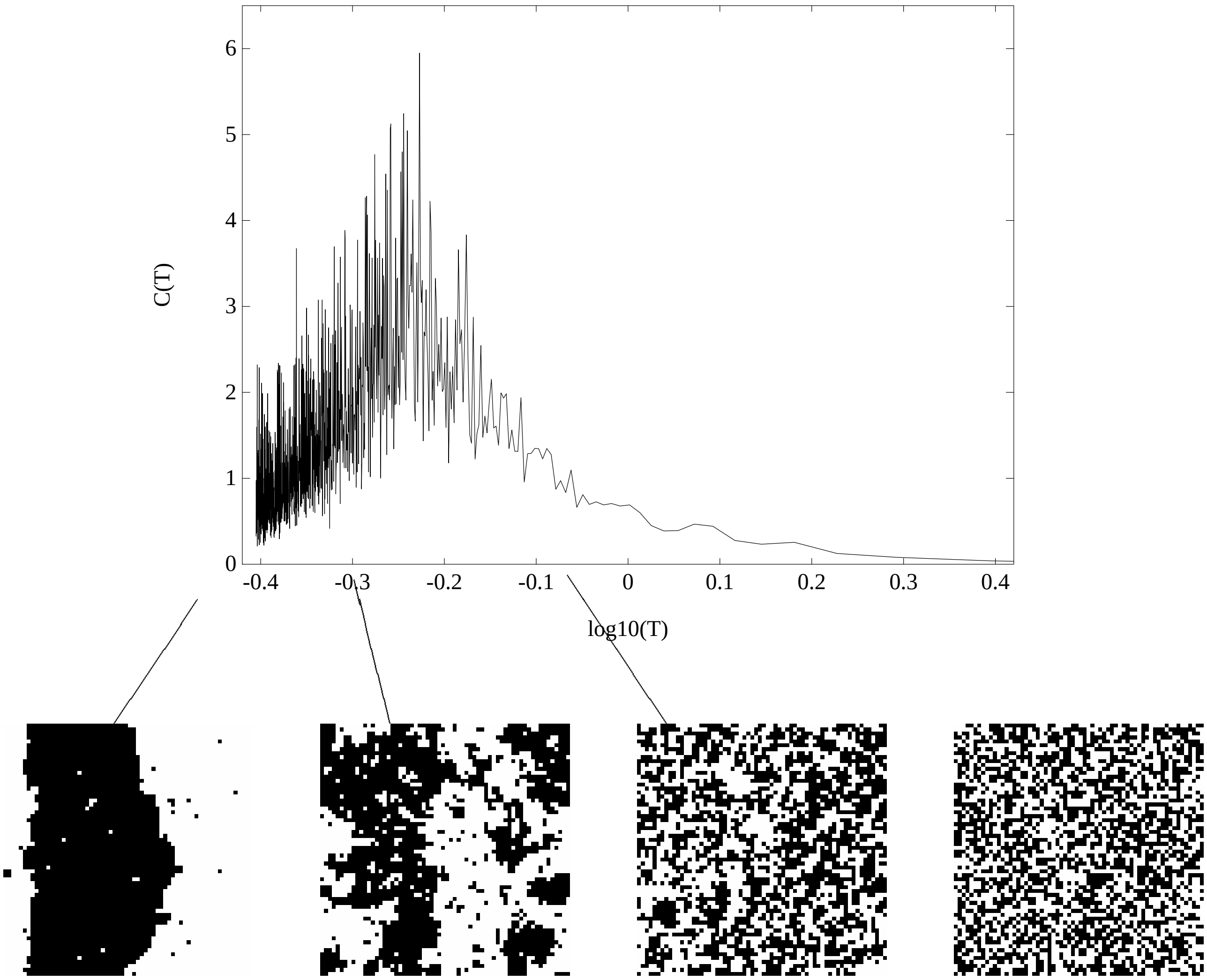
where $E_i(T)$ is the Gibbs energy at the current temperature and iteration, and i_{eq} is the number of iterations performed at each temperature when annealing is done.

We compared the effects of varying i_{eq} on measures of the specific heat and found that its peak characteristics changed, but its peak position was relatively constant. For the different i_{eq} in this example the resulting texture patterns were not noticeably affected. Representative texture examples for the case $i_{eq} = 10$ are shown in Figure 1. Each sample corresponds to i = 10, 100, 1000, and 10000 iterations and $\log_{10}(T)$ = 0.59, 0.054, -0.23, and -0.40 respectively. A peak is found at $T = T_* = 0.57$. A quick glance reveals that above T_* the pattern does not develop significantly, whereas below T_* a definite separation of black and white has occurred. A possible physical analogy to this behavior is that the system is a liquid at high temperature and a solid at low temperature. According to the interpretation of Kirkpatrick *et. al.* [1], this peaked region is where "the most optimization" occurs.

In Figure 2 we show the results of plotting specific heats for the isotropic attractive MRF when $n \geq 2$. Each specific heat data point was formed over 50,000 iterations of constant temperature texture synthesis for a wide variety of graylevels and temperatures. This is many more iterations than necessary for most cases, but was done to increase the probability that the systems were all in equilibrium (where equilibrium is defined as the energy being constant.) Plots are made from $T \in \{0.1, 0.27, 0.57, 1, 2.7, 10, 100, 1000, 10000\}$.

The specific heats, $C(T)$ shown here are all normalized by $(n-1)^4$ and plotted on a log-log scale for closer comparison. At high temperature the specific heat is minimum. For $n = 2, 4$ the energy variance is actually zero and $\log_{10} C(T)$ is not shown. Observe that for $n < 32$ the increase in $C(T)$ is linear from the minimum at the right to the first peak moving left. This linear behavior on the log-log plot indicates self-similarity. On the far left at cold temperature $C(T)$ drops. For $n = 16$ and $n = 32$ it is still high. Since equilibrium time grows with $(n-1)/T$, it is reasonable to expect that these points are not in equilibrium yet [7].

Notice that there is a strong peak at $\log_{10}(T) = -0.245$ for $n = 2$; this is consistent with the peak found for T_* in Figure 1 during annealing. Notice that this temperature



-0.40 -0.23 0.054 0.59

Figure 1: Isotropic textures corresponding to different temperatures, T. The normalized specific heat is plotted on a log-log scale above the samples; notice the changes in pattern relative to the peak in the specific heat. Each image is 64×64.

also appears to mark the left side of the peaked region for *all* the graylevels.

Notice that the right side of the peaks seems to move up in temperature with increasing n. The changing of a critical point by adding something (like more graylevels) to a solution is not new. The addition of salt to ice is a commonly occurring example in New England – the resulting mixture has a lower freezing point.

3.1.2 Aura matrix bandwidth

We have derived [15] particular structural changes which occur in the aura matrix, A, as the temperature of the MRF is lowered. At high temperature a texture configuration is usually random. The random pattern has an aura matrix with all entries nearly the same; it has "full" bandwidth, $n-1$. In contrast, under reasonable assumptions a minimum energy pattern when $\beta > 0$ will have a tridiagonal aura matrix with bandwidth of 1. In general, as the temperature drops so does the bandwidth. Similarly for the negative field ($\beta < 0$), the matrix tends to become anti-tridiagonal, or its "anti-bandwidth" decreases with temperature.

In Figure 3 we show results of plotting the aura matrix bandwidths vs. temperature.

3.2 Multiple transitions and bandwidth

Comparing Figure 3 with Figure 2 one can see that the peaked region of the specific heat coincides with the region of the bandwidth changes Of course there is no bandwidth change when $n = 2$ because that would mean the matrix becomes diagonal, which would mean that the lattice would have to break into two.

A natural question which arises is "are these places where the bandwidth changes also transition temperatures?" We first looked for finer resolution peaks in the specific heat corresponding to these transitions for many n and different annealing schedules, but it takes a very long time to simulate the MRF to equilibrium when the resolution is increased and we were unable to detect consistent peaks at all the bandwidth changes. Thus, it is unclear whether or not these points of change are analogous to critical temperatures for a multi-phase system.

Nonetheless, what is particularly interesting about these bandwidth changes is that they are correlated with distinct changes in the graylevel configurations of the textures. In particular, each time the bandwidth drops by 1, it can be interpreted as one more graylevel "precipitating" out of the original mixture of n graylevels. For example, at the first transition the color white stops mixing with black. (If the lattice were permitted to break, total separation would happen at this point.) For a texture of n graylevels there are $n-1$ bandwidth drops, and thus $n-1$ ways in which a color can "drop out" of the mixing.

Because this behavior is analogous to a solution of n substances where each precipitates out at as the temperature is lowered, it seems appropriate to call these "freezing points" transition temperatures. Thus, we have evidence for $n-1$ transition temperatures in the texture.

The precipitating behavior is illustrated for $n = 6$ in Figure 4. It can be seen that at each bandwidth decrease a new graylevel ceases to mix with black. The mixing and separation behaviors for an n graylevel texture are thus grouped into $n-1$ regions, separated by the temperatures of the bandwidth changes.

For texture synthesis at constant temperature as was done in [5], the inverse of the parameters $1/\beta_r$ can be treated as temperatures, and used to index into the types of patterns desired. Although we have only discussed the isotropic case in this paper, the anisotropic aura matrix can be formed as a sum of the isotropic ones over the different subneighborhoods as in (4) and a similar analysis can be carried out [7].

4 Conclusions

We have studied the effect of temperature on the textures modeled by an auto-binomial Markov/Gibbs random field. It has been shown that the structure in the texture aura matrix and hence in its co-occurrence matrices changes in a systematic way as the temperature parameter of the model is lowered. We have shown that there exist "transition temperatures" at which this matrix structure drops in bandwidth. To synthesize an equilibrium pattern with a particular bandwidth, one should use the temperature corresponding to that bandwidth. In addition, an analogy of the behavior of the model is made to the physical process of precipitation, whereby at each transition temperature a different graylevel separates from the mixture of the other graylevels. This is a natural extension of the crystallization analogy in simulated annealing to the case of multiple fluids and freezing points. We hope that this analogy is helpful in both understanding and controlling pattern formation with the auto-binomial MRF.

5 Acknowledgments

The authors would like to acknowledge I. M. Elfadel for proofreading, helpful comments, and early discussions on possible uses of the aura matrix.

References

[1] S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, "Optimization by simulated annealing," *Science*, vol. 220, no. 4598, pp. 671–680, 1983.

[2] J. L. Marroquin, *Probabilistic Solution of Inverse Problems.* PhD thesis, MIT, 1985.

[3] W. L. Masterton and E. J. Slowinski, *Chemical Principles.* W. B. Saunders Co., 1977.

[4] J. Besag, "Spatial interaction and the statistical analysis of lattice systems (with discussion)," *J. Roy. Stat. Soc., Ser. B*, vol. 36, pp. 192–236, 1974.

[5] G. R. Cross and A. K. Jain, "Markov random field texture models," *IEEE T. Patt. Analy. and Mach. Intell.*, vol. PAMI-5, no. 1, pp. 25–39, 1983.

[6] I. M. Elfadel and R. W. Picard, "Miscibility matrices explain the behavior of grayscale textures generated by Gibbs random fields," in *Proc. SPIE Conf. on Intell. Robots and Comp. Vis.*, vol. 1381, (Boston, MA), pp. 524–535, Nov. 1990.

[7] R. W. Picard, *Texture Modeling: Temperature Effects on Markov/Gibbs Random Fields.* ScD thesis, MIT, 1991.

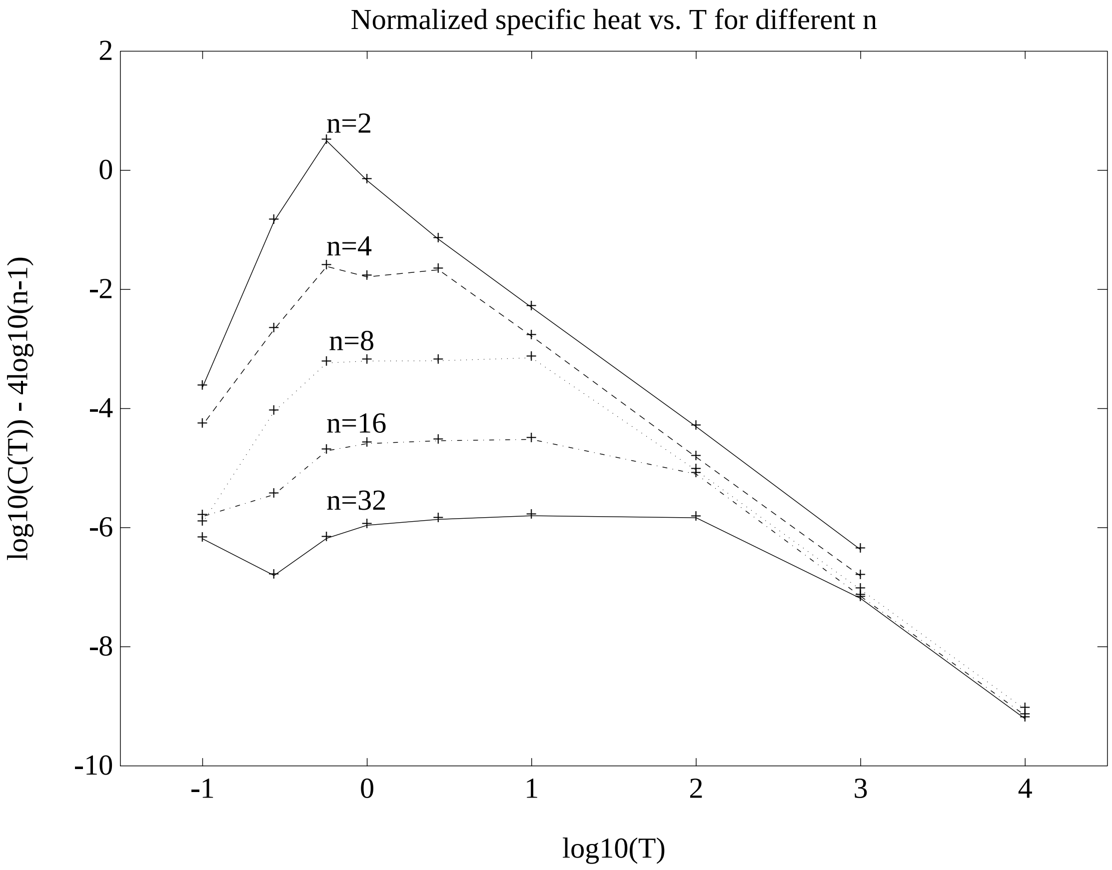


Figure 2: Normalized log of specific heat at each temperature for five different graylevels.

[8] W. M. Lawton and M. Lee, "Random field models for use in scene segmentation," tech. rep., Jet Propulsion Laboratory, March 1985.

[9] R. C. Dubes and A. K. Jain, "Random field models in image analysis," *J. of Appl. Stat.*, vol. 16, no. 2, pp. 131–163, 1989.

[10] P. Carnevali, L. Coletti, and S. Patarnello, "Image processing by simulated annealing," in *Proc. In Computer Vision* (M. A. Fischler and O. Firschein, eds.), Morgan Kaufmann, 1987.

[11] D. E. Van Den Bout and T. K. Miller, III, "Graph partitioning using annealed neural networks," *IEEE T. Neural Networks*, vol. 1, no. 2, pp. 192–203, 1990.

[12] D. K. Pickard, "Inference for discrete Markov fields: the simplest nontrivial case," *J. of the American Statistical Association*, vol. 82, no. 397, pp. 90–96, 1987.

[13] R. Kindermann and J. L. Snell, *Markov Random Fields and their Applications*. Providence, Rhode Island: American Mathematical Society, 1980.

[14] R. W. Picard, I. M. Elfadel, and A. P. Pentland, "Markov/Gibbs texture modeling: Aura matrices and temperature effects," in *Proc. IEEE Conf. on Computer Vision and Pattern Recognition*, (Maui, HI), pp. 371–377, June 1991.

[15] R. W. Picard and I. M. Elfadel, "On the structure of optimal aura matrices," Tech. Rep. 160, MIT Media Lab, Vision and Modeling, Feb. 1991.

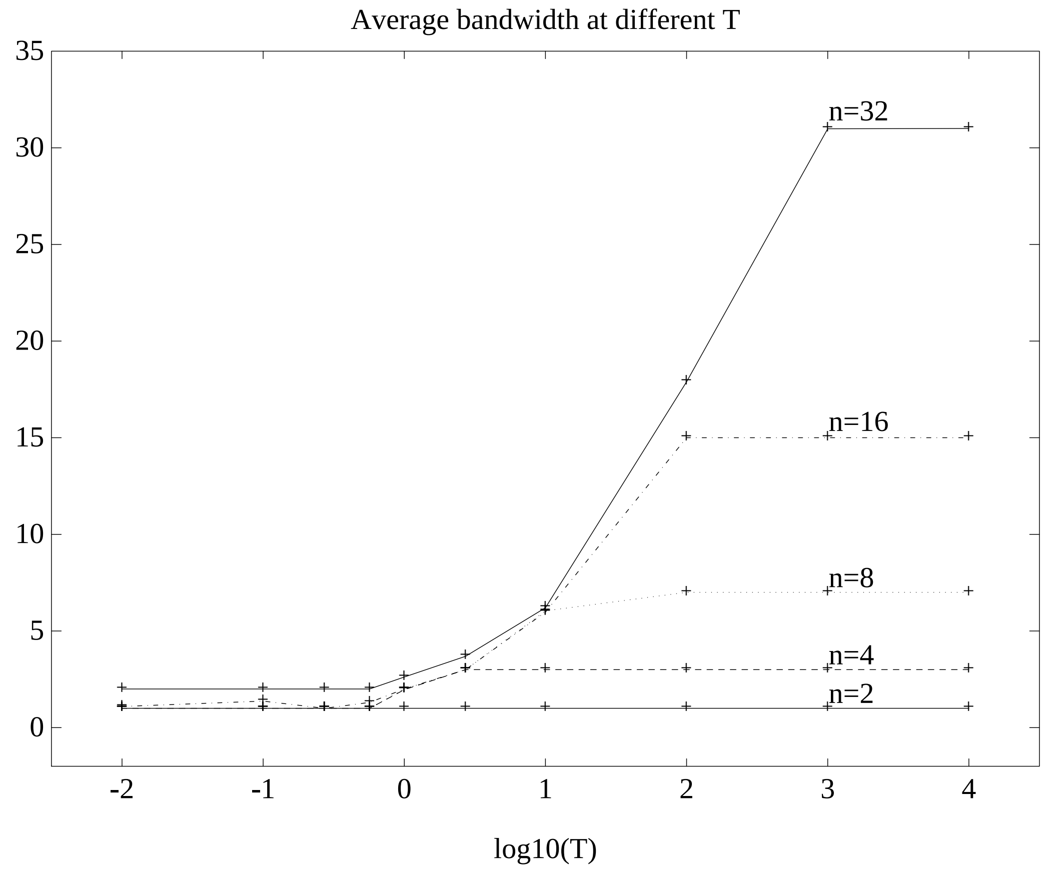
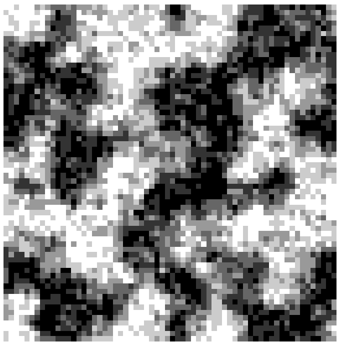
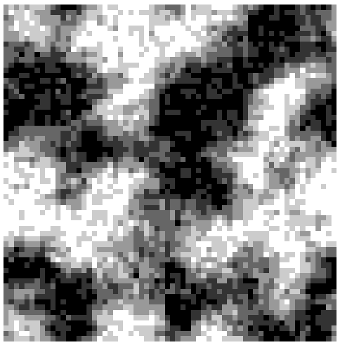


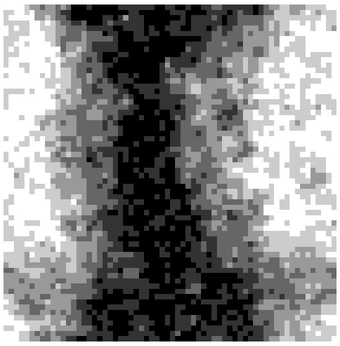
Figure 3: The bandwidths descend from n at high temperature to 1 at low temperature.



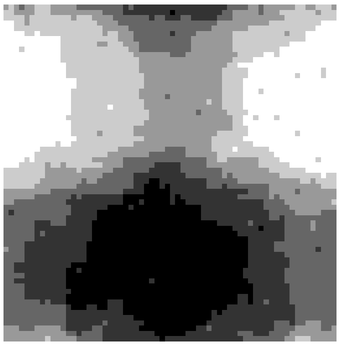
$$\mathbf{A} = \begin{bmatrix} 1454 & 859 & 345 & 60 & 10 & 0 \\ 859 & 908 & 685 & 229 & 42 & 5 \\ 345 & 685 & 848 & 596 & 201 & 53 \\ 60 & 229 & 596 & 870 & 632 & 341 \\ 10 & 42 & 201 & 632 & 974 & 869 \\ 0 & 5 & 53 & 341 & 869 & 1476 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} 1714 & 839 & 160 & 15 & 0 & 0 \\ 839 & 1104 & 658 & 124 & 3 & 0 \\ 160 & 658 & 1128 & 661 & 116 & 5 \\ 15 & 124 & 661 & 1136 & 631 & 161 \\ 0 & 3 & 116 & 631 & 1104 & 874 \\ 0 & 0 & 5 & 161 & 874 & 1704 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} 1868 & 822 & 37 & 1 & 0 & 0 \\ 822 & 1418 & 466 & 22 & 0 & 0 \\ 37 & 466 & 1550 & 649 & 25 & 1 \\ 1 & 22 & 649 & 1524 & 496 & 36 \\ 0 & 0 & 25 & 496 & 1440 & 767 \\ 0 & 0 & 1 & 36 & 767 & 1940 \end{bmatrix}$$



$$\mathbf{A} = \begin{bmatrix} 2546 & 182 & 0 & 0 & 0 & 0 \\ 182 & 2288 & 258 & 0 & 0 & 0 \\ 0 & 258 & 2248 & 222 & 0 & 0 \\ 0 & 0 & 222 & 2256 & 250 & 0 \\ 0 & 0 & 0 & 250 & 2292 & 186 \\ 0 & 0 & 0 & 0 & 186 & 2558 \end{bmatrix}$$

Figure 4: Patterns and aura matrices corresponding to a six graylevel MRF at transitions from bandwidth four down to bandwidth one. For example, bandwidth four with $m(0,5) = 0$ restricts black and white pixels from occuring in the same neighborhood.