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M-LATTICE: A NOVEL NON-LINEAR DYNAMICAL SYSTEM AND ITS APPLICATION TO HALFTONING

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ABSTRACT

This paper presents a novel non-linear dynamical system called the "M-lattice system". This system is rooted in the reaction-diffusion model, first proposed by Turing in 1952 to explain the formation of animal patterns such as zebra stripes and leopard spots. The M-lattice system is closely related to the analog Hopfield network and the cellular neural network, but has more flexibility in how its variables interact. In particular, the model is well-suited to a variety of applications formulated as constrained non-linear optimization. The present study demonstrates the use of this model for two different image halftoning examples. The first example synthesizes a halftone of Einstein in the "hand-drawn" style of the Wall Street Journal portraits; it illustrates how a more flexible quality metric can be used when the binary requirement is stated as an explicit constraint. The second example synthesizes halftones free of correlated artifacts; it illustrates the noise-shaping capability of the M-lattice system.

1. INTRODUCTION

The present research has originated in the investigation of the usefulness of reaction-diffusion systems for modeling natural textures. A reaction-diffusion system is a set of heat equations coupled by, typically non-linear, reaction terms. The reaction-diffusion model was first proposed by Turing in 1952 in order to explain mammal coat patterns, such as zebra stripes and leopard spots. Until recently, reaction-diffusion systems have been researched predominantly by mathematical biologists working on theories of natural pattern formation and by chemists working on modeling the dynamics of complex chemical reactions [1]. However, the past three years have seen a significant surge in interest in reaction-diffusion systems, primarily for exploiting them in the areas of computer graphics and image processing [2], [3], [4].

In order to form patterns a valid reaction-diffusion system must exhibit local instability to small random perturbations. That notwithstanding, the system should be stable in the large-signal sense for practical reasons. A major difficulty associated with the reaction-diffusion paradigm in its standard form is that the system is stable only for a restricted class of non-linear reaction functions. This drawback narrows the scope of the model's engineering applications, due to numerical overflow.

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A common approach aimed at preventing numerical overflow from plaguing the simulations of reaction-diffusion systems on the digital computer has been to clip the magnitudes of the state variables by adding an "if" statement to the numerical method (*e.g.*, Forward Euler) used for solving the system of differential equations [3]. However, this technique does not guarantee that the system will reach equilibrium; moreover, it destroys the mathematical integrity of the original dynamical system.

The main contribution of this paper is the formulation of the M-lattice system. By using a warping function to facilitate stability, this new system allows more flexible non-linear interactions than the reaction-diffusion system. Two of the capabilities of this model are illustrated in an application to digital halftoning of images.

2. M-LATTICE SYSTEM

Let $\psi_i(t) \in \Re$ be a state variable as a function of time at each lattice point i, where $i = 1, \ldots, N$. Let $\chi_i(t)$ be an output variable, obtained from $\psi_i(t)$ via $\chi_i(t) = g(\psi_i(t))$. Throughout this paper, the "warping" function, $g(u)$, an example of which appears in Figure 1, is a saturating piecewise linear function that can have an arbitrarily large number of straight-line segments. Construct $\vec{\psi}(t)$ and $\vec{\chi}(t)$ by concatenating $\psi_1(t), \ldots, \psi_N(t)$ and $\chi_1(t), \ldots, \chi_N(t)$, respectively into column vectors.

Definition 2.1 *Suppose that the given function, $\Phi(\vec{\chi}(t))$, is continuous, twice-differentiable, and bounded at least above. Let the matrix $\mathbf{A}$ be real, symmetric, and negative-definite: $\mathbf{A} \in \Re^{N \times N}$, $\mathbf{A} = [a_{ij}]$, $\mathbf{A} = \mathbf{A}^T$, and $\forall\ i\ \lambda_i[\mathbf{A}] < 0$. Then the M-lattice system is the following non-linear dynamical system:*

$$\frac{d\vec{\psi}(t)}{dt} = \mathbf{A}\vec{\psi}(t) + \vec{\nabla}_{\vec{\chi}}\Phi(\vec{\chi}(t)). \qquad (1)$$

As part of the analysis, we have shown that a subclass of the M-lattice system possesses asymptotic convergence properties, regardless of the initial conditions.

Proposition 2.1 *Consider a special case of the M-lattice system, (1), in which $\mathbf{A} = Diag\ \{a_1, \ldots, a_N\}$, $\forall\ i\ a_i < 0$. Any solution trajectory of this diagonal-state M-lattice system converges to a finite asymptotically stable fixed point, $\vec{\psi} \in \Re^N$ (or $\vec{\chi} \in [-1,\ 1]^N$).*

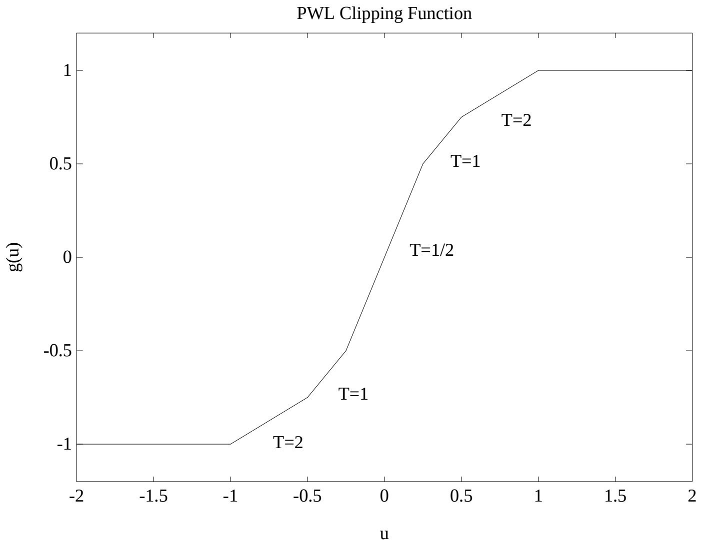


Figure 1: The plot of a 7-segment piece-wise linear clipping warping function. Slopes are $\frac{1}{T}$.

Thus far, no proof of total stability exists for the more general system, (1). However, if the system does not exhibit limit cycles and if there exists a fixed point of the form $\vec{\chi} \in \{-1,\ 1\}^N$, then convergence to such a fixed point is guaranteed [5]. In all cases of the type that we discuss below, the general M-lattice system exhibited convergence in computer simulation.

For certain types of objective functions, the M-lattice system converges to the (appropriately defined) local maxima of $\Phi(\vec{\chi}(t))$ with respect to $\vec{\chi}$ [6]. Thus, in many situations it is advantageous to use the M-lattice system for non-linear optimization. The next section describes two such examples.

3. HALFTONING

Suppose $\vec{n} \in \mathcal{Z}^2$; $s(\vec{n}) \in [-1,\ 1]$ is the continuous-tone (or finely quantized) original input image signal; $y(\vec{n}) \in \{-1,\ 1\}$ is the output halftone image; and $h(\vec{n})$ is a 2-D filter. The halftoning method must yield an image which appears perceptually similar to the original gray-scale image. Least-squares halftoning approaches receive continual attention, because they can employ explicit models of the human visual system and of the printing device [7].

3.1. HALFTONING AS NON-LINEAR PROGRAM

The problem of halftoning can be stated as a non-linear program:

$$\min_{\vec{y}} \frac{1}{2}\vec{y}^T\mathbf{B}\vec{y} - (\mathbf{B}\vec{s})^T\vec{y} \tag{2}$$

subject to constraints:

$$y_i^2 - 1 \quad \geq \quad 0, \tag{3}$$

where the vectors are the standard concatenations of the corresponding sequences, $\mathbf{B} = \mathbf{H}^T\mathbf{H}$, and $\mathbf{H}$ is a circulant matrix with $h(\vec{n})$ in the first row. The particular form of constraints, (3), forces each pixel to assume binary values.

In order to solve this problem using the M-lattice system we combine the objective function to be minimized, (2), with the N constraints, (3), into the Lagrangian cost functional with the help of the Karush-Kuhn-Tucker conditions [8]:

$$\min_{\vec{y}} \mathcal{L}(\vec{y}), \quad \text{where}$$

$$\mathcal{L}(\vec{y}) \quad = \quad \frac{1}{2}\vec{y}^T\mathbf{B}\vec{y} - (\mathbf{B}\vec{s})^T\vec{y} + \sum_i p_i(y_i^2 - 1), \tag{4}$$

$$p_i \quad \leq \quad 0, \quad p_i(y_i^2 - 1) = 0. \tag{5}$$

The Lagrange multipliers, p_i, are the varying penalty terms that enforce the constraints according to (5). As a result, the unconstrained minimization of $\mathcal{L}(\vec{y})$ in (4) produces the optimal halftone image.

The optimization problem, (4), is "programmed" onto the M-lattice system, (1), by setting $\vec{y}$ equal to $\vec{\chi}$, $\mathcal{L}(\vec{y})$ to $\Phi(\vec{\chi})$, and taking partial derivatives.

Halftoning with the Hopfield network requires setting $b_{ii} \geq 0$; otherwise, the optimal values of y_i will not be binary [9], [5]. However, treating halftoning as a non-linear programming problem and solving it with the M-lattice system offers considerable flexibility in the choice of the quality metric and in the functional form of constraints.

In order to demonstrate this flexibility, we encorporated orientation detection into the halftoning quality metric. The adaptive filter matrix, $\mathbf{H}$, was designed so as to include the information about the dominant orientation at each pixel of the original image, shown in Figure 2(a) [10]. The dominant orientation at a pixel is characterized by the angle, $\theta_i \in [-\pi,\ \pi]$, and by the relative strength (or magnitude), $m_i \in [0,\ 1]$, of that angle's presence at pixel i. Each filter is a 2-D Gaussian, whose level sets are oriented ellipses. Denote the diagonal matrix of variances by $\mathbf{V}_i$, the rotation matrix by $\mathbf{\Theta}_i$, and the position vector by $\vec{n} \in \mathcal{Z}^2$:

$$\mathbf{V}_i = \begin{bmatrix} \sigma_{i,x}^2 & 0 \\ 0 & \sigma_{i,y}^2 \end{bmatrix}, \mathbf{\Theta}_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{bmatrix}. \tag{6}$$

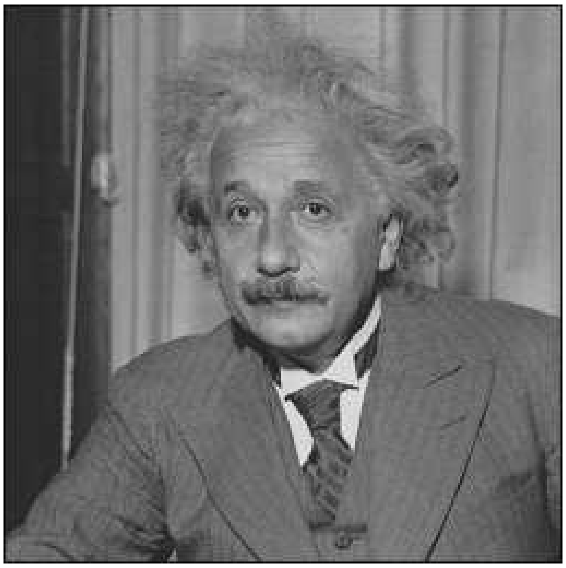
The relative sizes of $\sigma_{i,x}^2$ and $\sigma_{i,y}^2$ depend on m_i and determine the skewness of filters with respect to the dominant orientation:

$$\sigma_{i,y}^2 = \frac{L}{2}(1 - m_i), \quad \sigma_{i,x}^2 = L - \sigma_{i,y}^2, \tag{7}$$

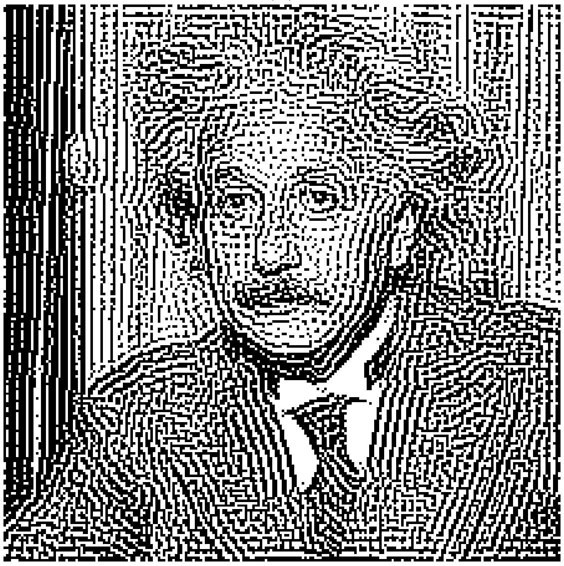
where $L \times L$ is the size of the filter mask in pixels. Then the (unnormalized) oriented low-pass filter is given by:

$$h_i(\vec{n}) \quad = \quad \exp\left\{-\vec{n}^T\mathbf{\Theta}_i^T\mathbf{V}_i\mathbf{\Theta}_i\vec{n}\right\}. \tag{8}$$

Since no effort is made to design $\mathbf{H}$ in a way that would result in $b_{ii} \geq 0$, the non-linear constraints provide the only mechanism for driving the output pixels to the limits of the gray scale. Figure 2 displays the result, which exhibits more of the line and curve features found in hand-drawn "halftones" (such as the Wall Street Journal portraits).



(a)



(b)

Figure 2: Orientation-sensitive halftoning. (a) the original "Einstein" image; (b) the "Einstein" image adaptively halftoned using orientation information at each pixel of the original.

3.2. NOISE-SHAPING LEAST-SQUARES HALFTONING

It is generally agreed that error diffusion produces the best results in terms of artifacts [11]. However, the causality of the algorithm prevents it from making sharp transitions and tracking edges properly [12]. In contrast, the least-squares halftoning techniques render edges well, but suffer from granular artifacts. We show that the M-lattice system naturally combines noise shaping with least-squares optimization, thereby offering the benefits of both.

Given a least-squares halftoning technique, set $\Phi(\vec{\chi})$ to the negative of the distortion measure. For example, if

$$\Phi(\vec{\chi}) \quad = \quad (\mathbf{H}^T\vec{s})^T\vec{\chi} - \frac{1}{2}\vec{\chi}^T\mathbf{H}^T\mathbf{H}\vec{\chi}, \tag{9}$$

then at an equilibrium (1) yields:

$$-\mathbf{A}\vec{\psi} \quad = \quad \mathbf{H}^T\vec{s} - \mathbf{H}^T\mathbf{H}\vec{\chi}, \text{ or} \tag{10}$$

$$-\left(\mathbf{A} - \mathbf{H}^T\mathbf{H}\right)\vec{\psi} \quad = \quad \mathbf{H}^T\vec{s} - \mathbf{H}^T\mathbf{H}(\vec{\chi} - \vec{\psi}). \tag{11}$$

Now set $-\left(\mathbf{A} - \mathbf{H}^T\mathbf{H}\right) = \mathbf{I}$, and let $\vec{q} \stackrel{\text{def}}{=} \vec{\chi} - \vec{\psi}$ be the quantization error (or the quantization noise) [5]. Then (10) and (11) become:

$$\vec{\psi} \quad = \quad \mathbf{H}^T\vec{s} - \mathbf{H}^T\mathbf{H}\vec{q}. \tag{12}$$

Thus, according to (12), the M-lattice system performs non-causal error diffusion in the steady-state limit. For perceptual reasons, it is desirable to minimize the low-frequency content of the quantization error. Since $\mathbf{H}^T\mathbf{H}$ is a smoothing filter, $\mathbf{H}_Q \stackrel{\text{def}}{=} \mathbf{I} - \mathbf{H}^T\mathbf{H}$ becomes a high-pass filter. Then it follows that $\mathbf{A} = -\mathbf{H}_Q$. The action of the high-pass noise shaping filter, $\mathbf{H}_Q$, gives the quantization noise the perceptually pleasant "blue" character [13]. We exploit the fact that $\mathbf{A}$ can have off-diagonal elements by making it act as a perceptually-based filter. Therefore, the resulting images correspond to local minima that are visually more pleasant than those produced using a diagonal $\mathbf{A}$ matrix.

Starting with the equation for error diffusion, (12), and reversing the above steps leads to (10), the equation for the M-lattice system in steady state. Error diffusion has been modeled as a Hopfield network that uses $q(\psi_i)$ in place of $g(\psi_i)$ [12]. However, the non-monotonicity of $q(\psi_i)$ causes instability. In contrast, slightly perturbing $\mathbf{A}$ so as to make it negative-definite guarantees that (1) will be stable for binary outputs. Hence, the M-lattice system is a more suitable model for non-causal error diffusion.

For the sake of simplicity we programmed the M-lattice system with the symmetric version of the original Floyd & Steinberg error filter. Figure 3 shows the magnified version of a test image and the result of halftoning it by the M-lattice system. The new method provides accurate detail rendition without introducing correlated texture. However, some perceptual artifacts still occur, because the filter coefficients have not yet been optimized after the causality constraint was lifted.

4. SUMMARY

We have presented the M-lattice system and applied it to digital halftoning of images as one of many potential ap-

(a)

(b)

Figure 3: Noise-Shaping Least-Squares Halftoning. (a) a portion of the original "Lenna" image (magnification is $\times 2$ on a side); (b) the image in (a) halftoned by the M-lattice system.

plications. As a non-linear programming technique, the M-lattice system is capable of solving constrained optimization problems with flexible objective functions. Orientation-sensitive halftoning makes use of this property. When the objective function is a quadratic form, the M-lattice system can be designed to perform blue noise filtering. This implies that the resulting halftone images can be made not only optimal in the least-squares sense, but also perceptually pleasant.

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5. REFERENCES

[1] A. M. Turing, "The chemical basis of morphogenesis," *Philosophical Transactions of Royal Society of London*, vol. 237, no. B, pp. 37–72, 1952.

[2] A. Witkin and M. Kass, "Reaction-diffusion textures," *Computer Graphics*, vol. 25, pp. 299–308, July 1991.

[3] G. Turk, "Generating textures on arbitrary surfaces using reaction-diffusion," *Computer Graphics*, vol. 25, pp. 289–298, July 1991.

[4] C. B. Price, P. Wambacq, and A. Oosterlinck, "Applications of reaction-diffusion equations to image processing," in *Third Int. Conf. on Image Proc. and its Applications*, (Warwick, UK), July 1989.

[5] A. Sherstinsky and R. W. Picard, "M-lattice system: Theory and potential applications," Tech. Rep. (In Preparation), M.I.T. Media Lab Perceptual Computing Group, 1993.

[6] M. Vidyasagar, "Minimum-seeking properties of analog neural networks with multilinear objective functions," Tech. Rep. (Unpublished), Bangalore, India, 1993.

[7] T. N. Pappas and D. L. Neuhoff, "Least-squares model-based halftoning," in *Proc. SPIE Conf. on Elect. Imaging Sci. and Tech.*, (San Jose, CA), Feb. 1992.

[8] M. S. Bazaraa and C. M. Shetty, *Nonlinear Programming*. New York: John Wiley and Sons, 1979.

[9] J. J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," *Proc. Nat. Acad. of Sci., USA*, vol. 81, pp. 3088–3092, May 1984.

[10] W. T. Freeman and E. H. Adelson, "The design and use of steerable filters," *IEEE Trans. on Pat. Anal. and Mach. Intell.*, vol. PAMI-13, pp. 891–906, Sept. 1991.

[11] R. W. Floyd and L. Steinberg, "An adaptive algorithm for spatial grey scale," in *SID Int. Sym. Dig. Tech. Papers*, pp. 36–37, 1975.

[12] D. Anastassiou, "Error diffusion coding for A/D conversion," *IEEE Trans. on Circ. and Sys.*, vol. 36, pp. 1175–1186, Sept. 1989.

[13] R. A. Ulichney, *Digital Halftoning*. Cambridge, MA: MIT Press, 1987.